A First Look at Tensor Diagrams

The Solution

Three Kinds of Matrix

$$[\text{point}] \times \mathbf{T} = [\text{point}]$$

$$[\text{point}] \times \mathbf{Q} = [\text{line}]^T$$

$$[\text{line}]^T \times \mathbf{Q}^* = [\text{point}]$$

Old Index Types

$$\mathbf{P} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

row

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

column

New Index Types

contravariant

$$\mathbf{P} = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}$$

covariant

Three Kinds of Matrix

$$T^i_j$$

Mixed

$$Q_{ij}$$

Pure covariant

$$\left(Q^*\right)^{ij}$$

Pure contravariant

The Multiplication Machine

$$\mathbf{P}\cdot\mathbf{L} = \begin{bmatrix} P_1 & P_2 & P_1 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$= P^1 L_1 + P^2 L_2 + P^3 L_3$$

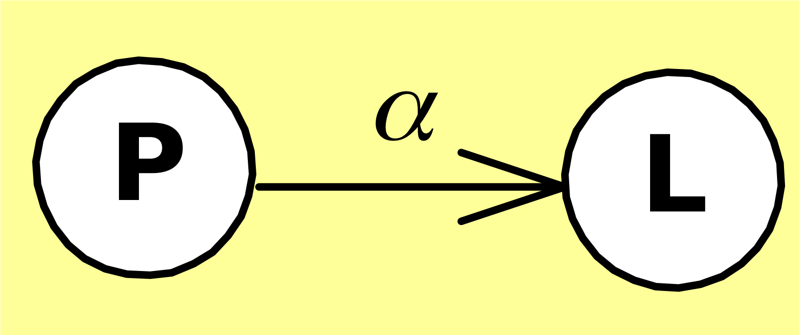
$$= \sum_i P^i L_i$$

$$= P^a E_a$$

Einstein Index Notation

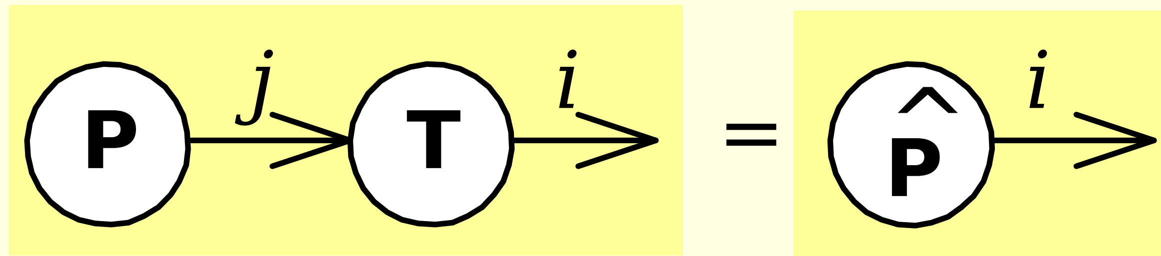
The Tensor Diagram of Dot Product

$$P^a L_a$$

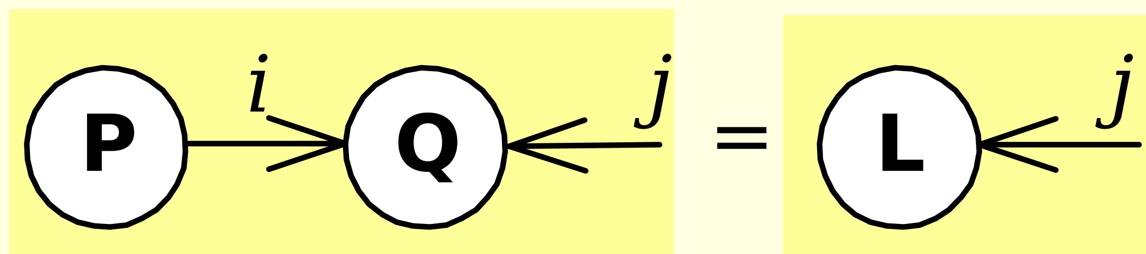


Three Kinds of Matrix

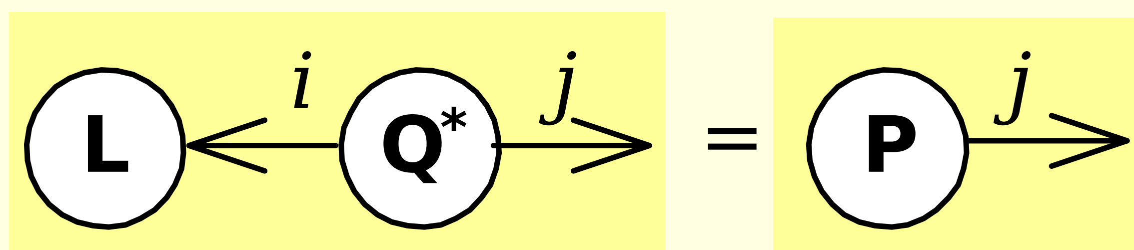
$$P^j T_j^i = \hat{P}^i$$



$$P^i Q_{ij} = L_j$$

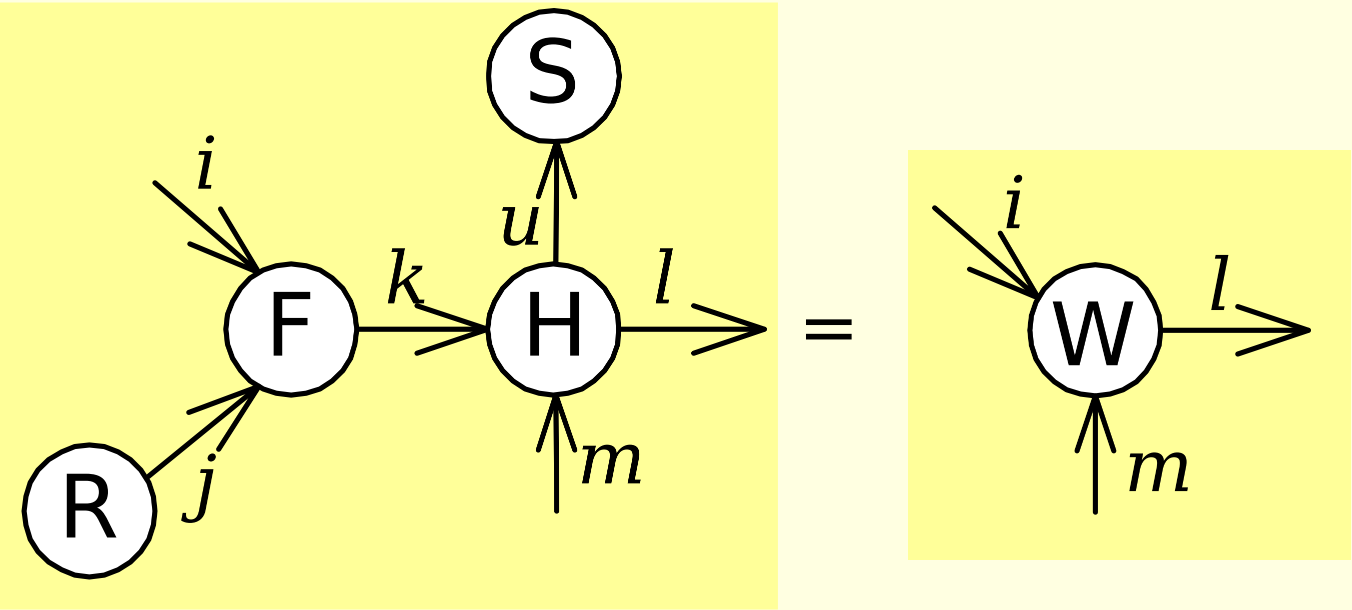


$$L_i \left(Q^*\right)^{ij} = P^j$$

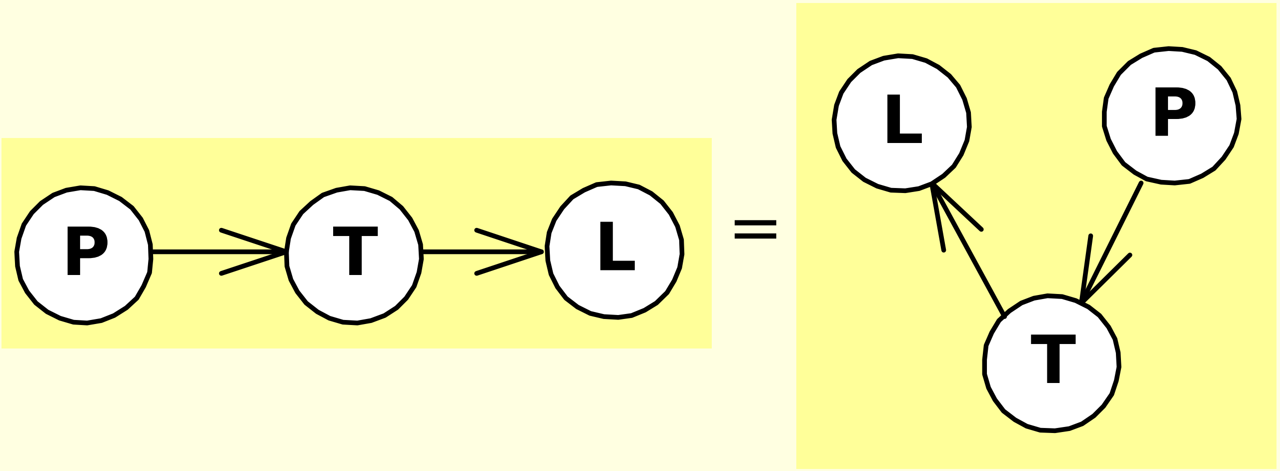


General Tensor Contraction

$$F_{ij}^{k} H_{km}^{lu} R^{j} S_{u} = W_{im}^{l}$$

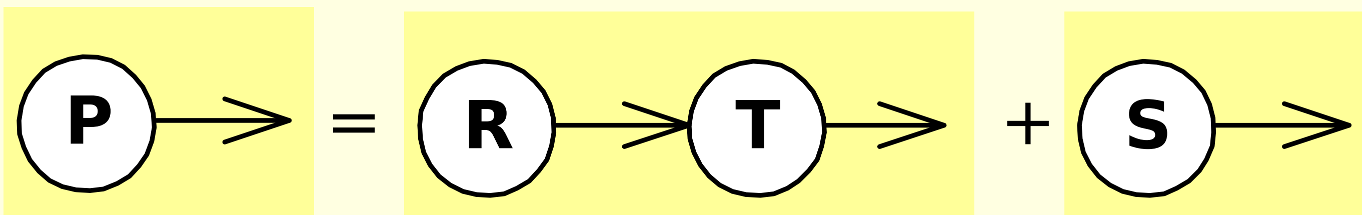


Rearranging Nodes Doesn’t Change Value



Sum of Terms

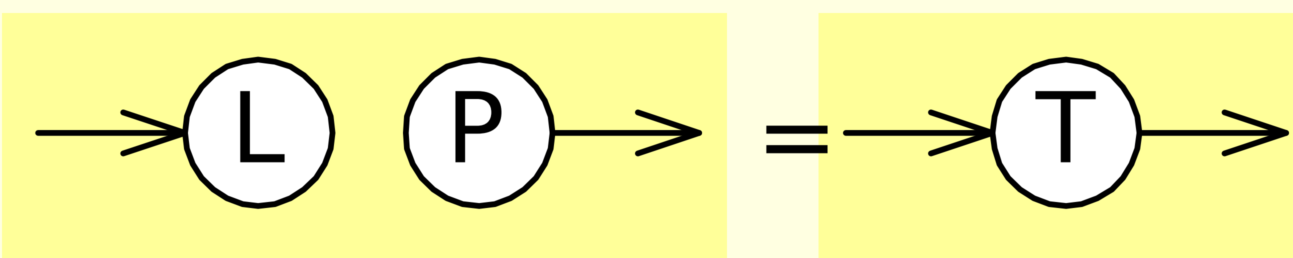
$$\mathbf{P} = \mathbf{R}\mathbf{T} + \mathbf{S}$$



Outer Product

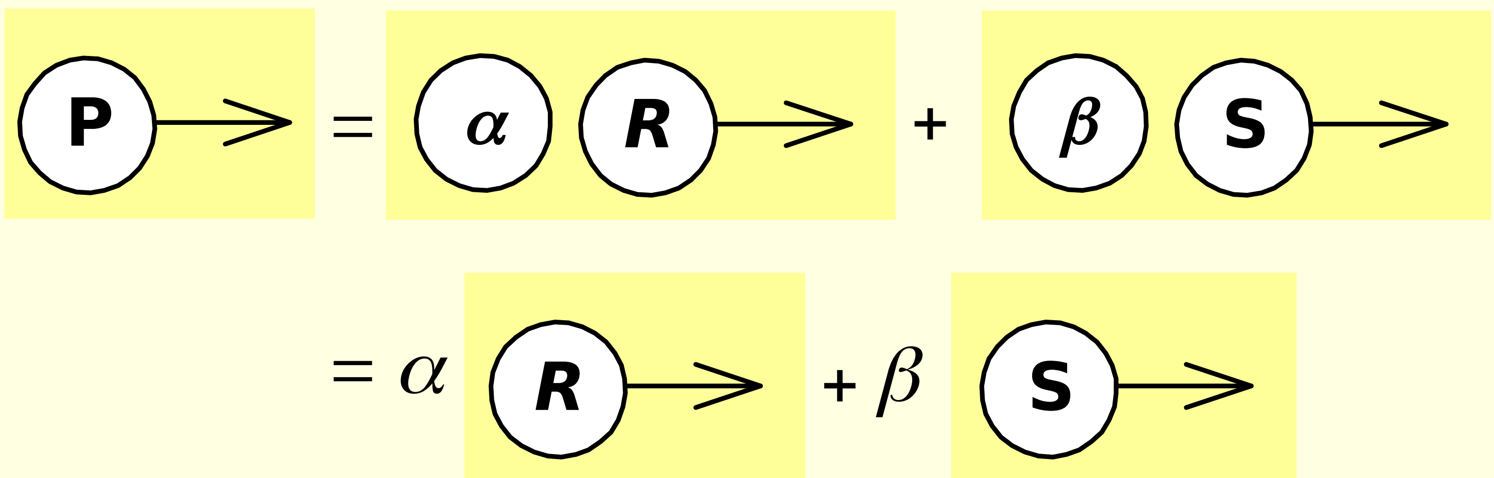
$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} ax & aw \\ bx & bw \end{bmatrix}$$

$$\mathbf{L}\,\mathbf{P} = \mathbf{T}$$



Scalar Product

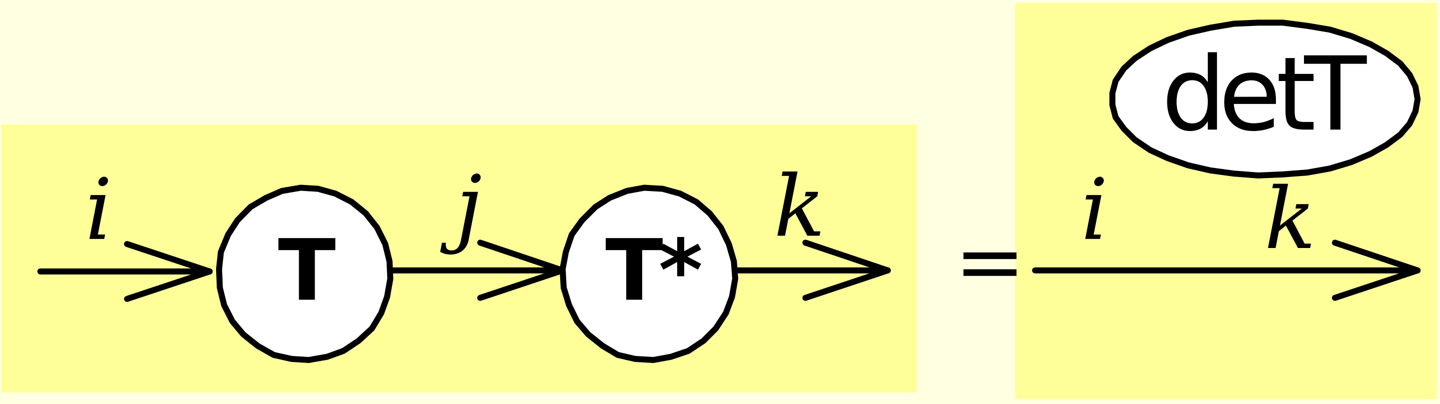
$$\mathbf{P} = a\mathbf{R} + b\mathbf{S}$$



Adjoint (of mixed tensor)

$$\mathbf{T}\mathbf{T}^{*} = (\det \mathbf{T})\,\mathbf{I}$$

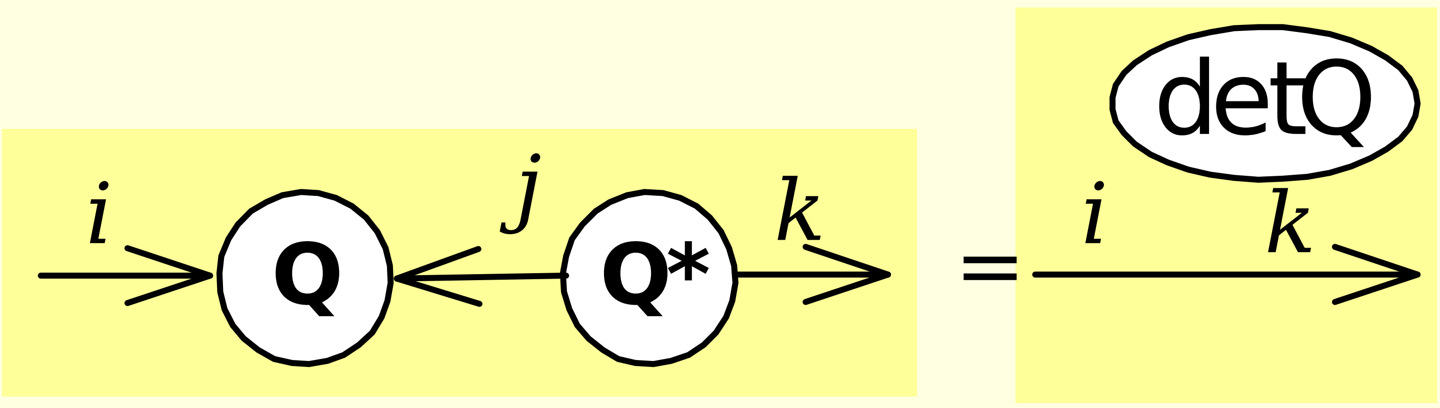
$$T_i^{\,j}\left(T^{*}\right)_j^{\,k} = (\det \mathbf{T})\,d_i^{\,k}$$



Adjoint (of covariant tensor)

$$\mathbf{Q}\mathbf{Q}^{*} = \left(\det \mathbf{Q}\right)\mathbf{I}$$

$$Q_{i,j}\left(Q^{*}\right)^{j,k} = \left(\det \mathbf{T}\right) d_{i}^{k}$$

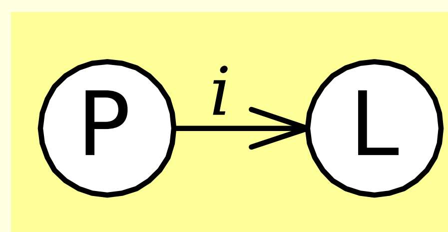


Point on a Line

$$ax + by + cw = 0 \qquad \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{L} = 0$$

$$P^i L_i = 0$$



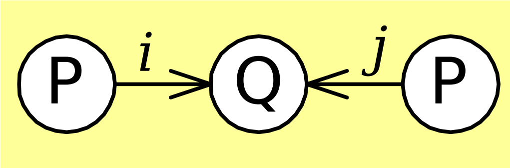
$= 0$

Point on a Quadratic Curve

$$
\begin{aligned}
Ax^2 + 2Bxy + 2Cxw \\
+ Dy^2 + 2Eyw \\
+ Fw^2 = 0
\end{aligned}
\qquad
\begin{bmatrix} x & y & w \end{bmatrix}
\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}
\begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$

$$\mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{P}^T = 0$$

$$P^i Q_{ij} P^j = 0$$



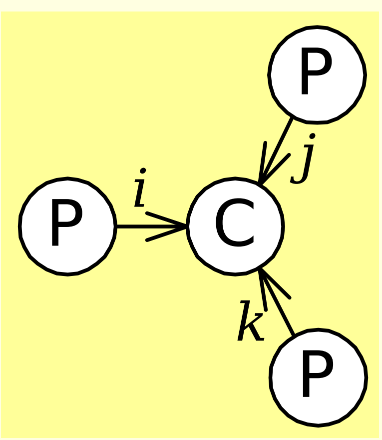
$= 0$

Point on a Cubic Curve

$$
\begin{aligned}
Ax^2 + 3Bx^2y + 3Cxy^2 + Dy^3 \\
+3Ex^2w + 6Fxyw + 3Gyw^2 \\
+2Hxw^2 + 3Jyw^2 \\
+Kw^2 = 0
\end{aligned}
$$

$$
\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$

$$P^i P^j P^k C_{ijk} = 0$$

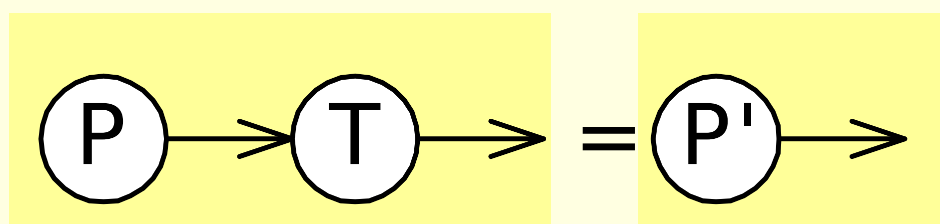


$= 0$

Transforming a Point

$$\mathbf{P}\mathbf{T} = \mathbf{P}'$$

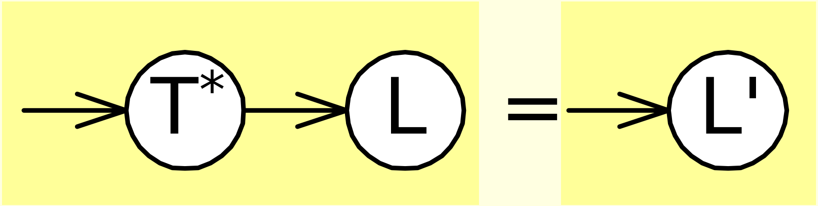
$$P^i T_i^j = \left(P' \right)^j$$



Transforming a Line

$$\left(\mathbf{T}^{*}\right)\mathbf{L} = \mathbf{L}'$$

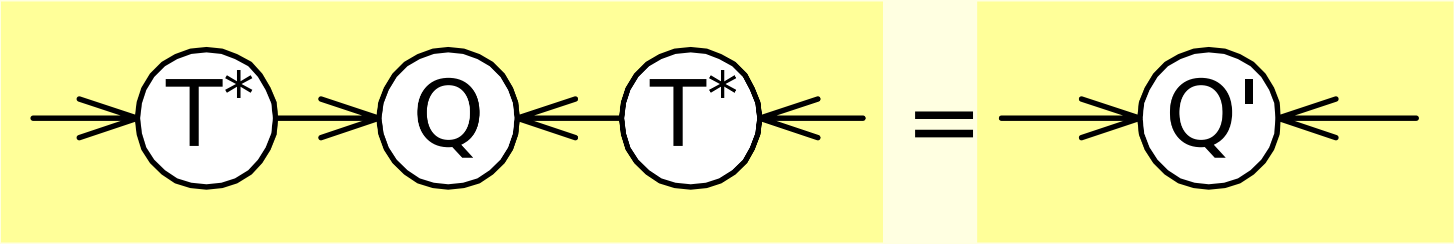
$$\left(T^{*}\right)^{i}_{j} L_{i} = \left(L'\right)_{j}$$



Transforming A Quadratic Curve

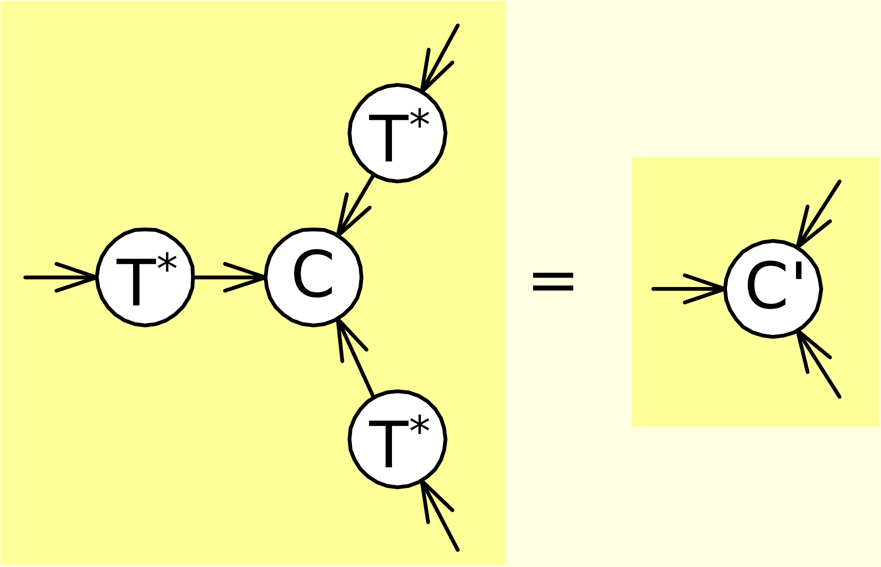
$$\left(\mathbf{T}^*\right)\mathbf{Q}\left(\mathbf{T}^*\right)^T = \mathbf{Q}'$$

$$\left(T^*\right)^i_k Q_{ij} \left(T^*\right)^j_l = \left(Q'\right)_{kl}$$



Transforming a Cubic Curve

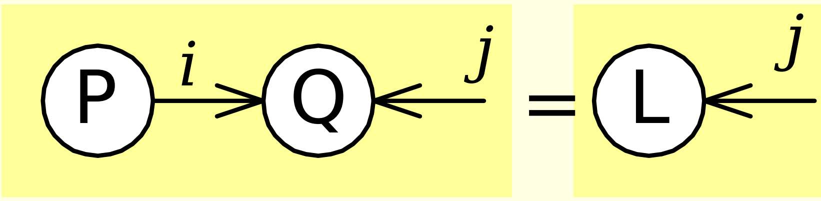
$$\left(T^*\right)_l^i \left(T^*\right)_m^j \left(T^*\right)_n^k C_{ijk} = \left(\hat{C}\right)_{lmn}$$



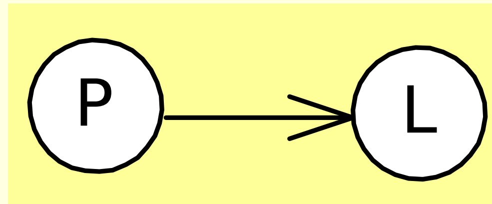
Tangent to Quadratic Curve

$$\mathbf{P}\times\mathbf{Q}=\mathbf{L}^T$$

$$P^iQ_{ij}=L_j$$



Dimensionality in Diagrams

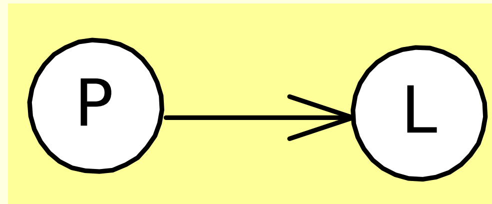


$$2D: \quad P^1 L_1 + P^2 L_2$$

$$3D: \quad P^1 L_1 + P^2 L_2 + P^3 L_3$$

$$4D: \quad P^1 L_1 + P^2 L_2 + P^3 L_3 + P^4 L_4$$

Dimensionality in Diagrams

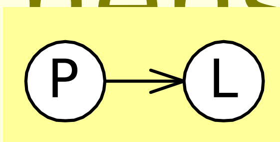


$$2D: \quad ax + bw$$

$$3D: \quad ax + by + cw$$

$$4D: \quad ax + by + cz + dw$$

Same Across Dimensionality



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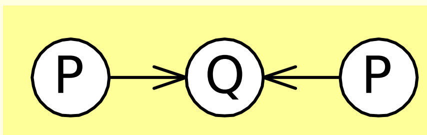

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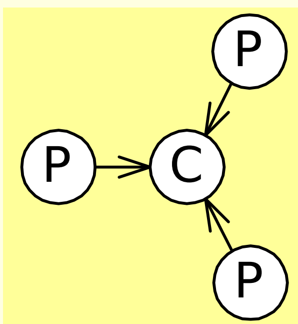

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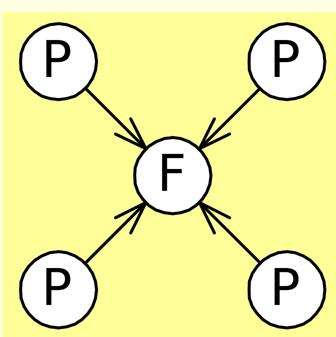

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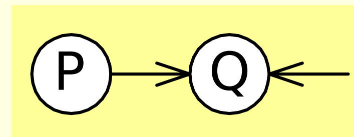

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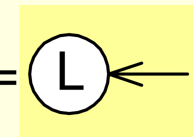

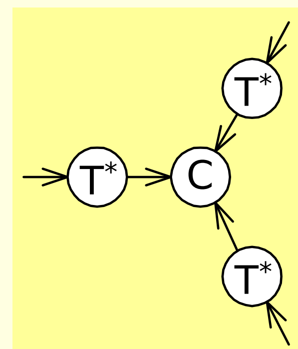

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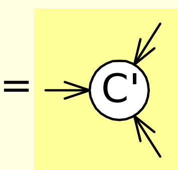


Changes with Dimensionality

- Cross Products
  - 3D (2DH)
  - 4D (3DH)
  - 2D (1DH)

3D (2DH) Levi-Civita Epsilon

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$

$$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$$

$$\varepsilon_{ijk} = 0 \quad \text{otherwise}$$

$$\varepsilon = \left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

3D (2DH) Cross Product

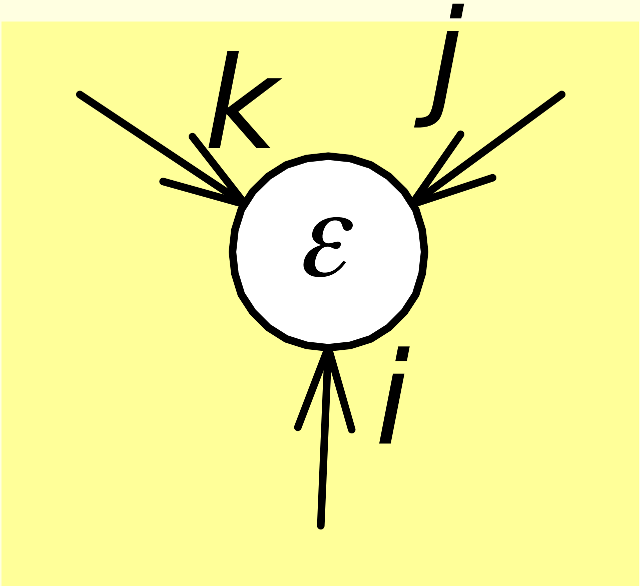
$$\begin{bmatrix} x_1 & y_1 & w_1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} =$$

$$\begin{bmatrix} y_1 w_2 - w_1 y_2 & w_1 x_2 - x_1 w_2 & y_1 x_2 - x_1 y_2 \end{bmatrix}$$

$$\left(P1\right)^i \left(P2\right)^j e_{ijk} = L_k$$

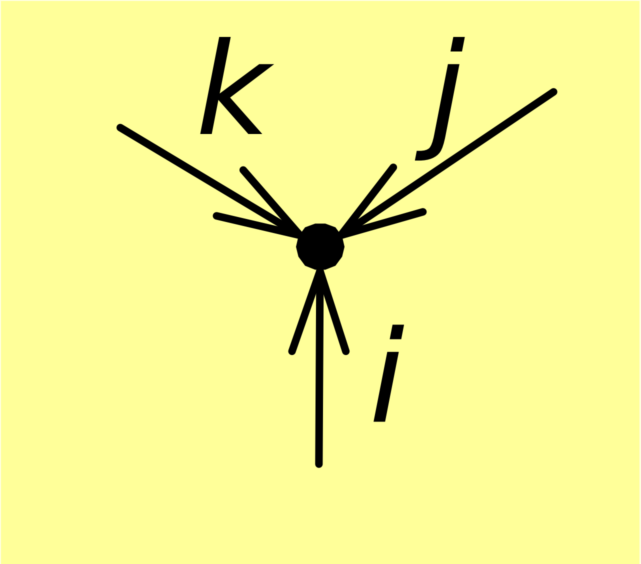
Levi-Civita Epsilon Diagram

$$e_{ijk}$$



Levi-Civita Epsilon Diagram

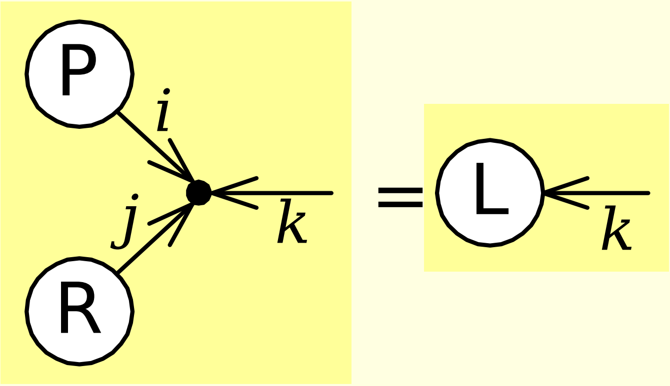
$$e_{ijk}$$



Cross Product

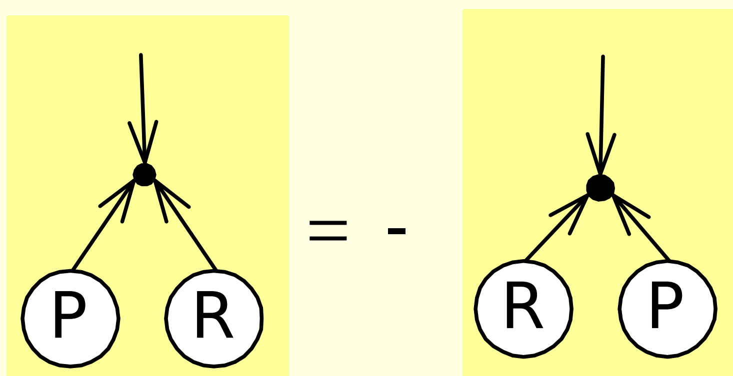
$$\begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix} \times \begin{bmatrix} R^1 & R^2 & R^3 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \qquad \mathbf{P} \times \mathbf{R} = \mathbf{L}$$

$$P^i R^j e_{ijk} = L_k$$



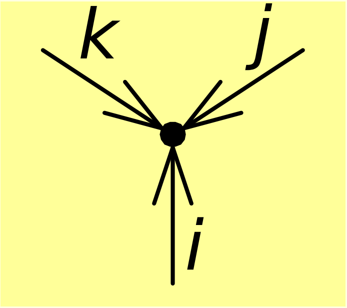
Anti-Symmetry and Epsilon

$$\mathbf{P \times R = - R \times P}$$

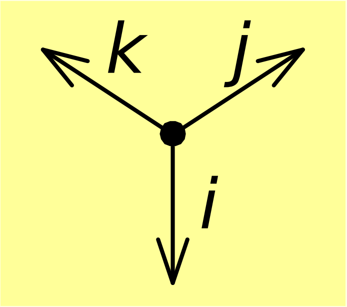


Levi-Civita Epsilon

COvariant e_{ijk}



CONTRAvari
ant e^{ijk}

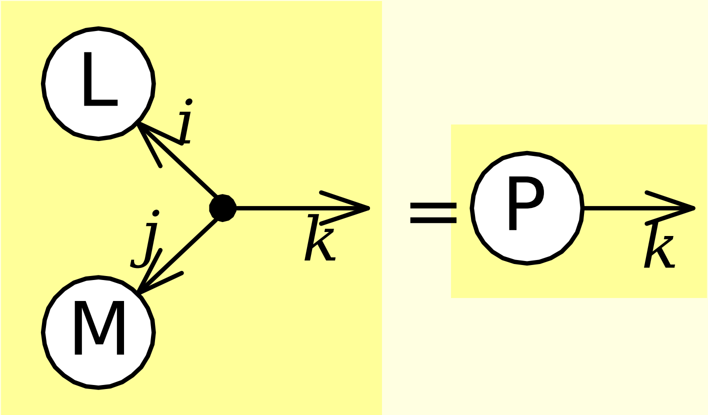


The Other Cross Product

$$\begin{bmatrix}L_1\\L_2\\L_3\end{bmatrix}\times\begin{bmatrix}M_1\\M_2\\M_3\end{bmatrix}=\begin{bmatrix}P^1 & P^2 & P^3\end{bmatrix}$$

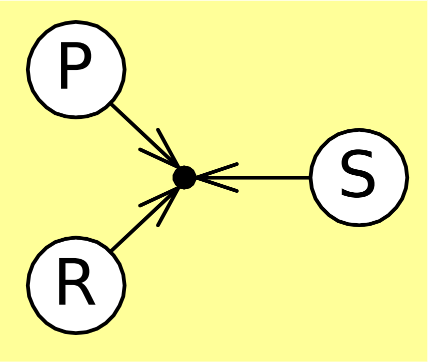
$$\mathbf{L}\times\mathbf{M}=\mathbf{P}$$

$$L_i M_j \epsilon^{ijk} = P^k$$

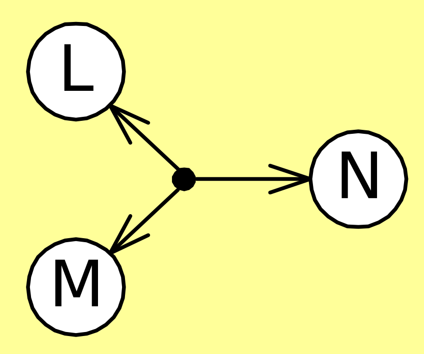


Triple Product

$$\mathbf{P}\ ´\ \mathbf{R}\times\mathbf{S} = \mathbf{R}\ ´\ \mathbf{S}\times\mathbf{P} = \mathbf{S}\ ´\ \mathbf{P}\times\mathbf{R}$$



$$\mathbf{L}\ ´\ \mathbf{M}\times\mathbf{N} = \mathbf{M}\ ´\ \mathbf{N}\times\mathbf{L} = \mathbf{N}\ ´\ \mathbf{L}\times\mathbf{M}$$

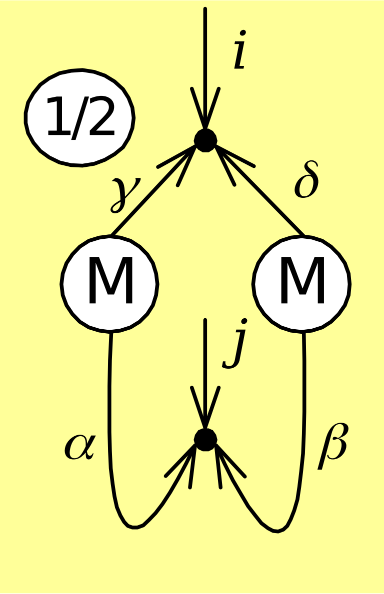


Adjoint of Matrix

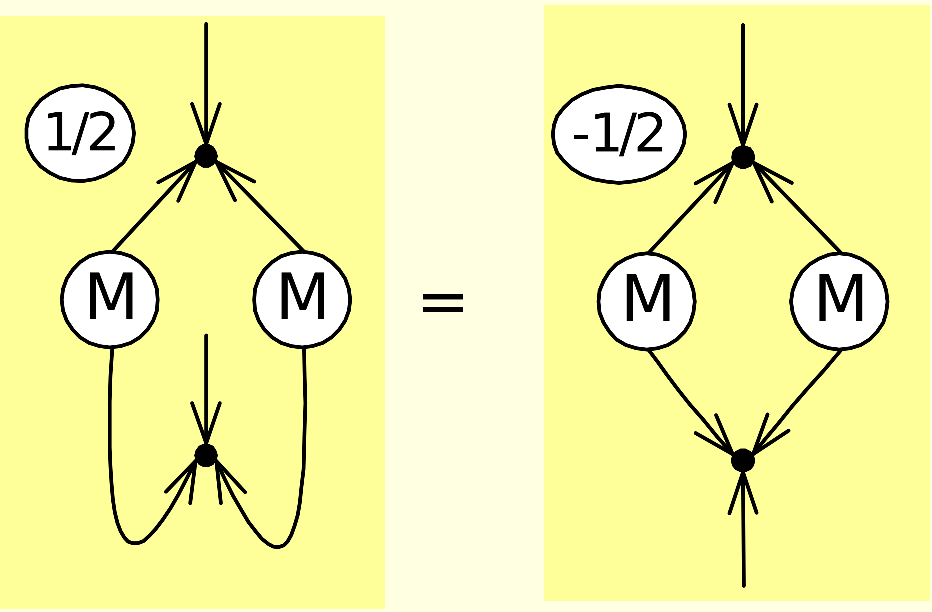
$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & M^{*23} & \cdot \end{bmatrix}$$

$$\left(M^{*}\right)^{23} = M_{21}M_{13} - M_{11}M_{23}$$

$$\left(M^{*}\right)^{ji} = \tfrac{1}{2}\varepsilon^{j\alpha\beta}\varepsilon^{i\gamma\delta}M_{\alpha\gamma}M_{\beta\delta}$$

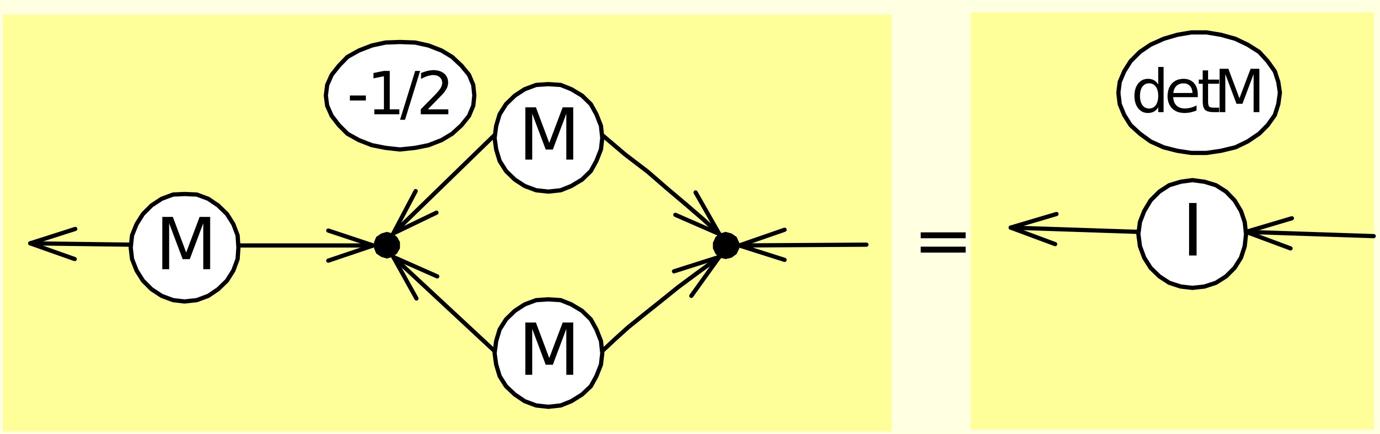


Adjoint of Matrix



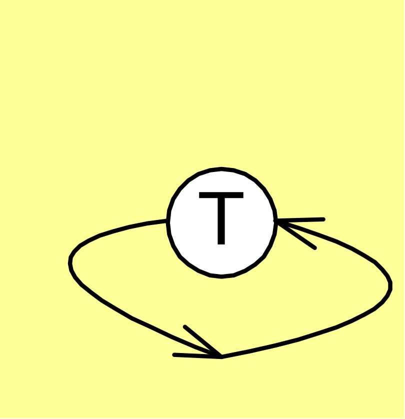
Determinant of Matrix

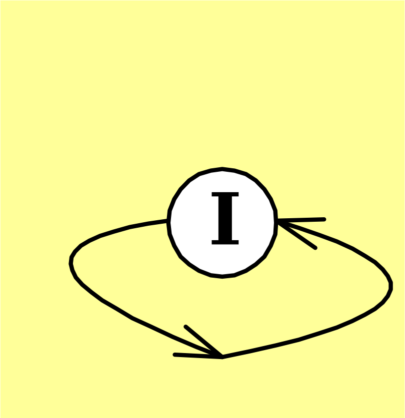
$$\mathbf{M}\mathbf{M}^{*} = \left(\det \mathbf{M}\right)\mathbf{I}$$



Trace of Matrix

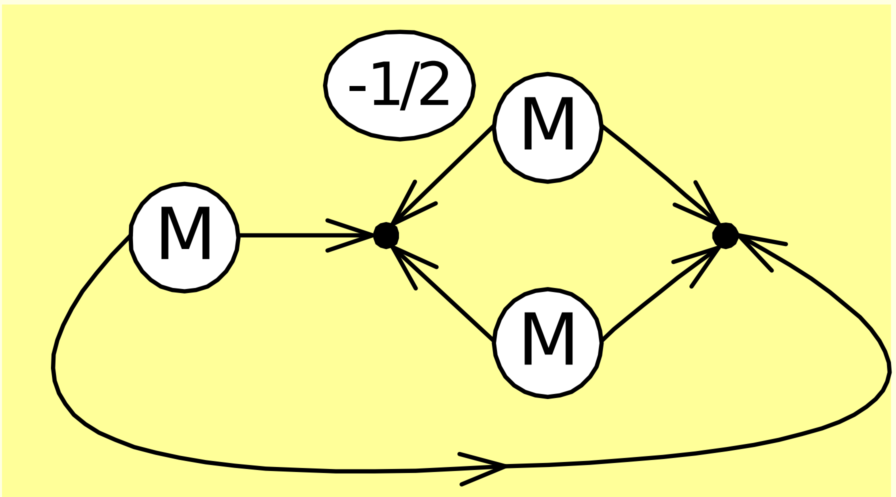
$$\operatorname{trace}\mathbf{T} = \sum_i T_i^i$$

trace T = 

trace $\mathbf{I}$ = 

$= 3$

Determinant of Matrix



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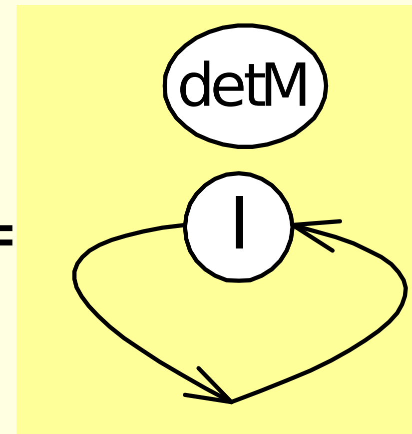

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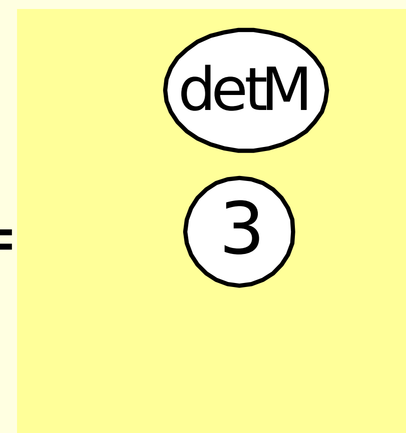
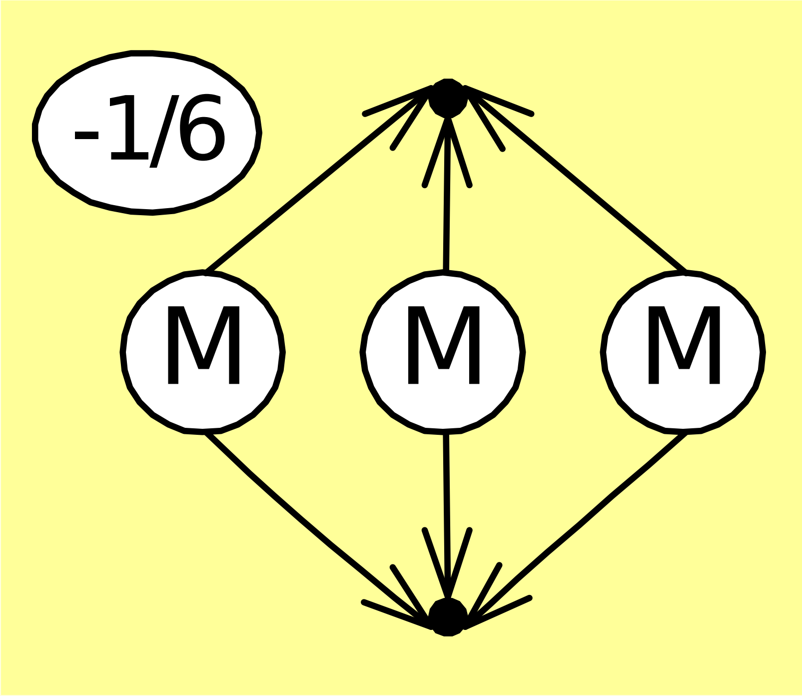


Determinant of Matrix



$$= \det \mathbf{M}$$